AN APPROXIMATE SOLUTION FOR A STEADY-STATE PROBLEM INVOLVING CONDITIONS OF CONVECTIVE HEAT TRANSFER

R. A. Pavlovskii

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We consider the mixed axisymmetric problem having boundary conditions of the III-rd kind which are specified at the surface of a disk. By means of a special method we find the approximate solution in analytical form and we evaluate its accuracy.

Let us consider the half-space $z > 0$, on portions of whose boundary—limited by a circle of unit radius—the conditions of convective heat transfer are satisfied, with constant temperature maintained over the remaining portions of the boundary plane. In the case of a homogeneous and isotropic medium, the determination of the temperature distribution in the half-space, as is well known, reduces to finding the harmonic function $\theta(r, z)$ for boundary conditions of the form*

$$\theta - k\frac{\partial\theta}{\partial z} = 1 \text{ when } z = 0,\ r < 1, \tag{1}$$

$$\theta = 0 \text{ when } z = 0,\ r > 1, \tag{2}$$

where $k = \text{const} > 0$.

In a number of papers (for example, [1–3]) this kind of problem is reduced to the solution of integral or integrodifferential equations. However, using the results obtained in these papers, we can find the solution for the problem only in numerical terms, which frequently limits the possibilities of its interpretation to a considerable extent. Let us therefore consider an approximate method of solving the stated problem; with this method we will be able to derive an analytical expression for the solution. **

Let us present the sought function $\theta(r, z)$ in the form of the sum of two functions

$$\theta = \theta' + \theta''$$

for $z = 0$ satisfying the following boundary conditions:

$$\theta' = \begin{cases} D, & r < c, \quad (3)\\ 0, & r > c, \quad (4)\end{cases}$$

$$\theta'' = f(r) = \begin{cases} -D, & 1 < r < c, \quad (5)\\ 0, & r > c, \quad (6)\end{cases}$$

$$\frac{\partial\theta''}{\partial z} = 0,\quad r < 1, \tag{7}$$

where D and c are parameters.

It is easy to see that boundary condition (2) is satisfied identically for any values of the parameters D and c. However, as regards boundary condition (1), we require that this condition be satisfied exactly, at least at two points: $r = 0$ and $r = 1$, i.e., we determine the parameters D and c from the simultaneous solution of the following two equations:

$$\theta''(0,\ 0) - k\frac{\partial\theta'(0, 0)}{\partial z} = 1 - D, \tag{8}$$

$$\theta''(1,\ 0) - k\frac{\partial\theta'(1, 0)}{\partial z} = 1 - D. \tag{9}$$

The problem of finding the functions θ' and θ'' involves no basic difficulties. The solution of the first of these is known in this case (see, for example, [5]) and is given by the formula

$$\frac{\theta}{D} = 1 - \frac{z}{\pi\sqrt{z^2 + (r+c)^2}}\left[\frac{a-c}{a+r}\,\Pi\left(\frac{\pi}{2},\ n_1,\ \nu\right) + \frac{a+c}{a-r}\,\Pi\left(\frac{\pi}{2},\ n_2,\ \nu\right)\right], \tag{10}$$

where

$$a = \sqrt{z^2 + r^2},\quad n_1 = -\frac{2r}{a+r},$$

$$n_2 = \frac{2r}{a-r},\quad \nu^2 = \frac{4cr}{(r+c)^2 + z^2}.$$

The following special expressions are derived from the general formula (10):

$$\theta'|_{r=0} = D\left(1 - \frac{z}{\sqrt{z^2 + c^2}}\right), \tag{11}$$

$$\left.\frac{\partial\theta'}{\partial z}\right|_{\substack{z=0\\ r<c}} = -\frac{2D}{\pi c}\,\frac{E\left(\frac{r}{c}\right)}{1 - \left(\frac{r}{c}\right)^2}, \tag{12}$$

$$\left.\frac{\partial\theta'}{\partial z}\right|_{\substack{z=0\\ r>c}} = -\frac{2D}{\pi r}\left[K\left(\frac{c}{r}\right) - \frac{E\left(\frac{c}{r}\right)}{1 - \left(\frac{c}{r}\right)^2}\right]. \tag{13}$$

The solution for the problem of finding θ'' will be sought in the form

$$\theta''(r,\ z) = \int_0^\infty A(\lambda)\,e^{-\lambda z} J_0(\lambda r)\,d\lambda. \tag{14}$$

Substitution of (14) into boundary conditions (5)–(7) leads to the following system of integral equations:

$$\int_0^\infty \lambda A(\lambda) J_0(\lambda r)\,d\lambda = 0,\quad r < 1, \tag{15}$$

$$\int_0^\infty A(\lambda) J_0(\lambda r)\,d\lambda = f(r),\quad r > 1. \tag{16}$$

We will present the function $f(r)$ in the form of the Hankel integral, assuming that $f(r) \equiv 0$ when $r < 1$:

*In the formulation and solution of the problem, for the sake of generality, we will use dimensionless quantities.

**An approximate solution for an analogous problem with reverse specification of the boundary conditions has been derived in [4].

$$f(r) = \int_0^\infty \lambda \bar{f}(\lambda) J_0(\lambda r)\, d\lambda, \tag{17}$$

where

$$\bar{f}(\lambda) = \int_0^\infty r J_0(\lambda r) f(r)\, dr. \tag{18}$$

Recalling (5) and (6), we have

$$\bar{f}(\lambda) = -D \int_1^c r J_0(\lambda r)\, dr = \frac{D}{\lambda}\,[J_1(\lambda) - cJ_1(\lambda c)]. \tag{19}$$

With the aid of (17) and (19), it is not difficult to bring system (15)–(16) to the form

$$\int_0^\infty \lambda B(\lambda) J_0(\lambda r)\, d\lambda = f_1(r), \quad r < 1, \tag{20}$$

$$\int_0^\infty B(\lambda) J_0(\lambda r)\, d\lambda = 0, \quad r > 1, \tag{21}$$

where

$$B(\lambda) = A(\lambda) - D\,[J_1(\lambda) - cJ_1(\lambda c)], \tag{22}$$

$$f_1(r) = D \int_0^\infty \lambda J_0(\lambda r)\,[cJ_1(\lambda c) - J_1(\lambda)]\, d\lambda. \tag{23}$$

The solution of system (20)–(21) is known [3] and is given by the formula

$$B(\lambda) = \int_0^1 \varphi(t) \sin \lambda t\, dt, \tag{24}$$

where

$$\varphi(t) = \frac{2}{\pi} \int_0^t \frac{\rho f_1(\rho)\, d\rho}{\sqrt{t^2 - \rho^2}}. \tag{25}$$

Having substituted (23) into (25), having altered the sequence of integration, and using the familiar integral expressions of the Bessel functions [6], we obtain

$$\varphi(t) = \frac{2Dt}{\pi} \left(\frac{1}{\sqrt{c^2 - t^2}} - \frac{1}{\sqrt{1 - t^2}} \right). \tag{26}$$

Using the integral representation of the function $J_1(\lambda)$, by means of (22), (24), and (26) we find the solution for the system of paired integral equations (15)–(16):

$$A(\lambda) = \frac{2D}{\pi} \int_0^1 \frac{t \sin \lambda t\, dt}{\sqrt{c^2 - t^2}} - cDJ_1(\lambda c) \tag{27}$$

thus simultaneously finding the solution to the problem of determining the function θ''

$$\theta''(r, z) = \frac{2D}{\pi} \int_0^1 \frac{t\, dt}{\sqrt{c^2 - t^2}} \int_0^\infty \exp(-\lambda z) J_0(\lambda r) \sin \lambda t\, d\lambda - cD \int_0^\infty \exp(-\lambda z) J_0(\lambda r) J_1(\lambda c)\, d\lambda. \tag{28}$$

Simple transformations and integration yields the formula for the function θ'' at both the boundary and the axis of symmetry of the system:

$$\theta''|_{r=0} = D \left[\frac{z}{\sqrt{z^2 + 1}} - 1 + \frac{2}{\pi} \left(\arcsin \left(\frac{1}{c} \right) - \frac{z}{\sqrt{c^2 + z^2}} \operatorname{arctg} \frac{\sqrt{c^2 + z^2}}{z \cdot \sqrt{c^2 - 1}} \right) \right], \tag{29}$$

$$\theta''\Big|_{\substack{z=0 \\ r<1}} = D \left(\frac{1}{\pi} \arcsin \frac{2 - c^2 - r^2}{c^2 - r^2} - \frac{1}{2} \right). \tag{30}$$

From formulas (11), (12), (29), and (30) we have the expressions for the temperature and its normal derivative:

$$\theta(0, z) = \frac{2D}{\pi} \left[\arcsin \left(\frac{1}{c} \right) - \frac{z}{\sqrt{c^2 + z^2}} \operatorname{arctg} \frac{\sqrt{c^2 + z^2}}{z \sqrt{c^2 - 1}} \right], \tag{31}$$

$$\theta(r, 0)|_{r<1} = D \left(\frac{1}{2} + \frac{1}{\pi} \arcsin \frac{2 - c^2 - r^2}{c^2 - r^2} \right), \tag{32}$$

$$\frac{\partial \theta(r, 0)}{\partial z}\Big|_{r<1} = \frac{\partial \theta'(r, 0)}{\partial z}\Big|_{r<1<c} \quad \text{[see (12)]}. \tag{33}$$

Proceeding from (33) and (12), for the over-all heat flow of the system we have

$$Q = -2\pi \int_0^1 r \frac{\partial \theta(r, 0)}{\partial z}\, dr = \frac{4D}{c} \int_0^1 \frac{rE\left(\frac{r}{c}\right)}{1 - \left(\frac{r}{c}\right)^2}\, dr = 4Dc \left[K\left(\frac{1}{c}\right) - E\left(\frac{1}{c}\right) \right].^* \tag{34}$$

All of the above-derived formulas contain the previously unknown parameters D and c which we determine—proceeding from the system of equations (8)–(9) —by substituting (12) and (30) into that system when r = 0 and r = 1. By means of appropriate transformations we derive the relationships which associate these parameters with the specified quantity k:

$$k = \frac{\pi c \left(1 - \frac{1}{c^2}\right) \left(\frac{1}{2} + \frac{1}{\pi} \arcsin \frac{2 - c^2}{c^2} \right)}{2E\left(\frac{1}{c}\right) - \pi \left(1 - \frac{1}{c^2}\right)}, \tag{35}$$

$$D = \left(\frac{1}{2} + \frac{k}{c} + \frac{1}{\pi} \arcsin \frac{2 - c^2}{c^2} \right)^{-1}. \tag{36}$$

It is not difficult to establish that

$$\lim_{c \to 1} k = 0, \quad \lim_{c \to 1} D = 1,$$
$$\lim_{c \to \infty} k = \infty, \quad \lim_{c \to \infty} D = 0.$$

Thus, the derived approximate solution is valid for any value of $k \in [0, \infty)$.

The curves showing the variation in the parameters c and D as functions of k are shown in Fig. 1. By means of these curves, on the basis of the specified Biot number (k = 1/Bi), we can find the values of the parameters c and D, and then, using the above-derived formulas, calculate both the field distribution and the integral characteristics of the system (the heat flow and the thermal resistance).

*Formula (34) may also be used for the determination of the thermal resistance.

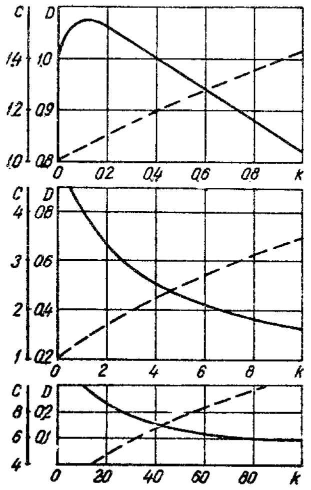

Fig. 1. Graph for determining parameters c (dashed line) and D (solid curve).

We would be particularly interested in evaluating the level of nonuniformity in the distribution of the heat flow over the disk as a function of k. Such an evaluation is possible if we employ the formula which follows from (12) with consideration of (33):

$$\delta q=\frac{\dfrac{\partial\theta(1,\,0)}{\partial z}-\dfrac{\partial\theta(0,\,0)}{\partial z}}{\dfrac{\partial\theta(1,\,0)}{\partial z}+\dfrac{\partial\theta(0,\,0)}{\partial z}}=\frac{\dfrac{2}{\pi}E\left(\dfrac{1}{c}\right)+\dfrac{1}{c^2}-1}{\dfrac{2}{\pi}E\left(\dfrac{1}{c}\right)-\dfrac{1}{c^2}+1}.$$

The curves showing the relative nonuniformity of heat-flow distribution as a function of k are shown in Fig. 2. An examination of these curves shows that when $k > 3$ the distribution of the heat flow can be regarded as uniform, with an error not exceeding several percent.

The accuracy of this approximate solution is determined by the deviation of the boundary condition for $r < 1$ from the specified equation (1):

$$\alpha(r)=1-\theta(r,\,0)+k\,\frac{\partial\theta(r,\,0)}{\partial z},\quad r\in[0,\,1].^{*}\tag{37}$$

Relationship (37), after substituting (32) and (33) into it, with consideration of (12), leads to the formula

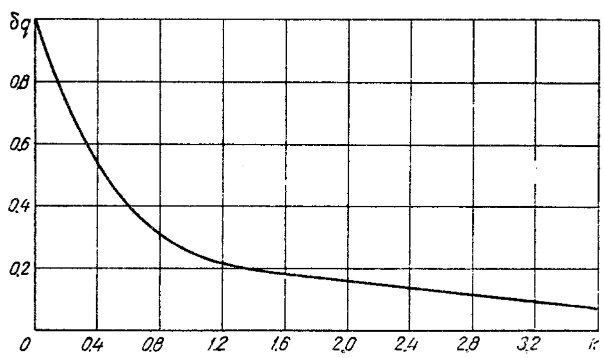

Fig. 2. Relative nonuniformity of heat-flux distribution on disk.

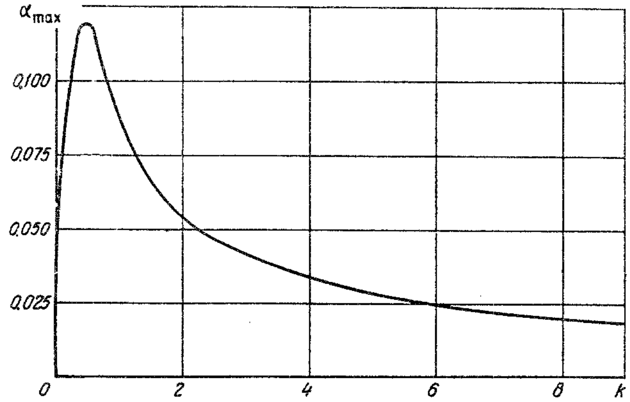

Fig. 3. Discrepancy between exact and approximate boundary conditions at r < 1.

$$\alpha(r)=D\left[\frac{1}{2}-\frac{1}{D}+\frac{1}{\pi}\arcsin\frac{2-c^2-r^2}{c^2-r^2}+\frac{2k}{c\pi\left(1-\dfrac{r^2}{c^2}\right)}E\left(\frac{r}{c}\right)\right].\tag{37'}$$

Calculation of the quantity $\alpha(r)$ showed that for any $k\in[0,\infty)$ it does not exceed 0.12 (for $k\sim 1$) and that it is distributed nonuniformly along the disk radius. The curve showing the change in the absolute magnitude of the maximum values of $\alpha(r)$ as a function of k is shown in Fig. 3. Bearing in mind that $\alpha(r)$ reaches its maximum values over a relatively small interval of variation in the radius, we can maintain that the error in the derived approximate solution for any k does not exceed several percent of the maximum value of the function at the boundary.

In conclusion, it should be noted that the above-cited result can also be used for similar analogous problems from potential theory, and in particular in the calculation of the electric field of linearpolarization electrodes.

NOTATION

θ is the temperature; Q is the heat-flux; δq is the relative nonuniformity of heat-flux distribution; Bi is the Biot number; A, B, f, f_1, and φ are the function symbols; α is the discrepancy of boundary conditions; r, and z, are the cylindrical coordinates; ρ and t are the integration variables; λ is the parameter of variable separation; c and D are the parameters; J_0 and J_1 are the Bessel functions of the first kind, of zero and first order, respectively; K, E, and Π are the total elliptic integrals of the first, second, and third kinds, respectively.

REFERENCES

1. G. V. Poddubnyi, IFZh, 4, no. 5, 1961.
2. E. T. Artypaev, IFZh, 7, no. 10, 1964.
3. Yu. N. Kuz'min, ZhTF, 36, no. 2, 1966.

*In particular, when $k = 0$ ($c = 1, D = 1$) we have $\alpha(r) = 0$, i.e., in this case the derived solution is exact.

4. R. A. Pavlovskii, IFZh [Journal of Engineering Physics], 14, no. 2, 1968.

5. N. N. Mirolyubov et al., Methods of Calculating Electrostatic Fields [in Russian], Vysshaya shkola, 1963.

6. I. S. Gradshtein and I. M. Ryzhik, Tables of Integrals, Sums, Series, and Products [in Russian], Fizmatgiz, 1962.

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